arXiv:1010.2249v1 [math-ph] 11 Oct 2010

Clebsch–Gordan Coefficients of the Quaternion Group

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(Dated: October 13, 2010)

The Clebsch–Gordan coefficients of the Kronecker products of the irreducible representations of the Quaternion Group Q_8, of the Generalized Quaternion Groups Q_{16} and Q_{32}, and of the factor group $(Q_{32} \times Q_{32})/\{(1,1),(-1,-1)\}$ are computed as eigenvectors of a well-known matrix of triple-products of the irreducible representations.

PACS numbers: 03.65.-wi, 02.20.Hja
Keywords: Quaternion, finite group, Clebsch–Gordan, representation

I. SCOPE

The quaternion group Q_8 is a non-abelian group of order 8 [1–3]. In relativistic quantum-mechanics, the quaternion algebra appears as a representation of Dirac bi-spinors [4]; the direct product $Q_8 \times Q_8$ spans the algebra of the 2-fermion matrix elements [5].

Whereas the Clebsch–Gordan (CG) decomposition of other small groups is well documented serving studies of point group operations in molecular and solid-state physics [6, 7], the CG series of Q_8—although simple—seems not to be readily available. The manuscript follows standard concepts to reduce the product of irreducible representations of some quaternionic groups of low order [8].

II. STRUCTURE OF Q_8

A. Multiplication Table, Classes

Q_8 is the fourth group of order 8 in the GAP4 library [9–12], the fifth in the Schaps enumeration [13]. It is called $\Gamma_2 a_2$ by Hall–Senior [14], 8/5 in the Thomas-Wood enumeration [15], and apparently this index carries over as 8.5 [16]. The $|Q_8| = 8$ elements g_i are enumerated with their standard symbols in Table I.

Table II shows the index l of the product $g_j g_k = g_l$ at the crossing of row j and column k. In the hypercomplex notation it is summarized as

$$\mathtt{-1}^2 = \mathtt{1};\ \mathtt{i}^2 = \mathtt{j}^2 = \mathtt{k}^2 = \mathtt{-1};\ \mathtt{i}\cdot\mathtt{j} = \mathtt{k};\ \mathtt{j}\cdot\mathtt{i} = \mathtt{-k}; \quad (1)$$

and so on [17]. Fig. 1 is another view, a digraph of 8 vertices, one per group element [18]. Blue edges lead from label j to label l if $g_j g_5 = g_l$; read edges lead from label j to label l if $g_j g_3 = g_l$. (The set of generators $\{g_5, g_3\}$ here is only one out of many choices. Equivalent ambivalences govern the Cayley graphs further down.)

The 5 conjugacy classes in Q_8 are

$$\mathcal{C}_1 = \{g_1\}, \mathcal{C}_2 = \{g_2\}, \quad (2)$$
$$\mathcal{C}_3 = \{g_3, g_4\}, \mathcal{C}_4 = \{g_5, g_6\}, \mathcal{C}_5 = \{g_7, g_8\}. \quad (3)$$

Table III enumerates the 4 irreducible representations of dimension 1 and 1 of dimension 2 [19]. The 1-dimensional representations are already part of the character table, and the 2-dimensional representation may be

TABLE I: Indices of the Q_8 group elements g_j, their standard names, and orders.

j	1	2	3	4	5	6	7	8
g_j	`1`	`-1`	`i`	`-i`	`j`	`-j`	`k`	`-k`
	1	2	4	4	4	4	4	4

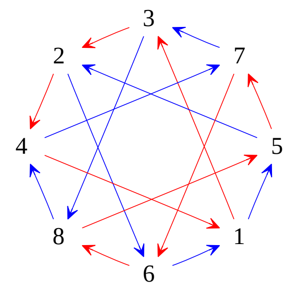

FIG. 1: A Cayley graph of Q_8.

TABLE II: Cayley (multiplication) table of Q_8.

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	2	1	8	7	5	6
4	3	1	2	7	8	6	5
5	6	7	8	2	1	4	3
6	5	8	7	1	2	3	4
7	8	6	5	3	4	2	1
8	7	5	6	4	3	1	2

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TABLE III: Characters of the representations $R^{(\alpha)}$ of the five Q_8 classes (3)

\ $\mathcal{C}$	1	2	3	4	5
$R^{(1)}$	1	1	1	1	1
$R^{(2)}$	1	1	1	-1	-1
$R^{(3)}$	1	1	-1	1	-1
$R^{(4)}$	1	1	-1	-1	1
$R^{(5)}$	2	-2	0	0	0

chosen as

$$g_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad g_2 \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{4}$$

$$g_3 = -g_4 \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad g_5 = -g_6 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \tag{5}$$

$$g_7 = -g_8 \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{6}$$

or as the Pauli matrices.

B. CG coefficients

The Kronecker product of two 1-dimensional representations is obviously 1-dimensional, irreducible and defines CG coefficients (CGS's) which are all equal to zero or one:

$$R^{(1)} \otimes R^{(\alpha)} = R^{(\alpha)} \quad (\forall\alpha); \quad R^{(2)} \otimes R^{(3)} = R^{(4)}; \tag{7}$$

$$R^2 \otimes R^{(4)} = R^{(3)}; \quad R^{(3)} \otimes R^{(4)} = R^{(2)}. \tag{8}$$

The remaining case is the CG series of the square of the 2-dimensional representation,

$$R^{(5)} \otimes R^{(5)} = R^{(1)} \dotplus R^{(2)} \dotplus R^{(3)} \dotplus R^{(4)}. \tag{9}$$

(Real-valued 4-dimensional representations like this one are often preferred in mechanics [20].) The multiplicities m_γ

$$R^{(\alpha)} \otimes R^{(\beta)} \equiv \dot{\sum_\gamma} m_\gamma R^{(\gamma)} \tag{10}$$

have been calculated from the traces (characters) χ of the associated representations via the standard relation [21, (3.20)] [22, (7.22)]

$$m_\gamma = \frac{1}{|Q_8|} \sum_g \chi_{\alpha\otimes\beta}(g) \overline{\chi_\gamma(g)}, \tag{11}$$

where the overbar denotes complex-conjugation. (These Kronecker products are called *simply* reducible because no multiplicity m_γ is larger than one [23].)

The Clebsch–Gordan coefficients $\langle \alpha i, \beta k | \gamma l\rangle$ are the elements of the square matrix which provides the similarity transformation of the matrix $R^{(\alpha\otimes\beta)}$ to block diagonal form [24]. Its row and column dimension is $n_\alpha n_\beta$, the product of the dimensions of the matrices $R^{(\alpha)}$ and $R^{(\beta)}$. An implicit representation is [25, 26][22, (7.34b)]

TABLE IV: CG matrix of $R^{(5)} \otimes R^{(5)}$ of Q_8.

j	k	$R^{(1)}$	$R^{(2)}$	$R^{(3)}$	$R^{(4)}$
1	1	0	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
1	2	$-1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	1	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	$1/\sqrt{2}$	$1/\sqrt{2}$

$$\langle \alpha j, \beta k|\gamma l\rangle \overline{\langle \alpha j', \beta k'|\gamma l'\rangle} \propto \frac{1}{|Q_8|} \sum_g R^{(\alpha)}_{j,j'}(g) R^{(\beta)}_{k,k'}(g) \overline{R^{(\gamma)}_{l,l'}(g)}. \tag{12}$$

As written here, the summation over m_γ replicas of the irreducible representation $R^{(\gamma)}$ is not yet carried out. In the tables IV, VII, VIII and so on further below, the column double-indices j and k refer to the representations in the (essentially arbitrary) ordering $\alpha \le \beta$ of the representations. The indices l are added to header rows if the dimensions n_γ are larger than one.

We regard this equation as a matrix equation between two matrices with rows represented by the multi-index j, k, l and columns represented by the multi-index j', k', l', i.e, matrices with row and column dimension which are the square of the dimension of the CG matrix. Given the elements of the right hand side in terms of the elements of the elements of the irreducible representations R, the left hand side is the spectral representation known as Mercer's theorem [27]. The CG matrices are the eigenvectors of the matrix seen on the right hand side associated with the non-vanishing eigenvalues [28].

All CG matrices tabulated in this manuscript are unitary.

III. QUATERNION GROUP Q_{16}

The product table of the $|Q_{16}| = 16$ elements of Q_{16} is reproduced in Table V. The upper left corner is the Q_8 subgroup, a copy of Table II. (The subgroup structure is discussed by Bohanon and Reid [29].) The Hall–Senior number is 16.14.

Fig. 2 is a digraph with vertices labeled by the indices of the group elements. Blue edges point from j to l if $g_j g_9 = g_l$; red edges point from j to l if $g_j g_5 = g_l$.

The 7 conjugacy classes are

$$\mathcal{C}_1 = \{g_1\},\, \mathcal{C}_2 = \{g_2\},\, \mathcal{C}_3 = \{g_3, g_4\},\, \mathcal{C}_4 = \{g_5, \ldots, g_8\}, \tag{13}$$

$$\mathcal{C}_5 = \{g_9, \ldots, g_{12}\},\, \mathcal{C}_6 = \{g_{13}, g_{15}\},\, \mathcal{C}_7 = \{g_{14}, g_{16}\}, \tag{14}$$

with elements of order 1 in $\mathcal{C}_1$, order 2 in $\mathcal{C}_2$, order 4 in $\mathcal{C}_3$–$\mathcal{C}_5$, and order 8 in $\mathcal{C}_6$ and $\mathcal{C}_7$. There are 7 irreducible

TABLE V: Cayley table of Q_{16}

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	4	2	1	8	7	5	6	12	11	9	10	15	16	14	13
4	3	1	2	7	8	6	5	11	12	10	9	16	15	13	14
5	6	7	8	2	1	4	3	15	16	14	13	11	12	10	9
6	5	8	7	1	2	3	4	16	15	13	14	12	11	9	10
7	8	6	5	3	4	2	1	13	14	15	16	10	9	12	11
8	7	5	6	4	3	1	2	14	13	16	15	9	10	11	12
9	10	11	12	13	14	15	16	2	1	4	3	6	5	8	7
10	9	12	11	14	13	16	15	1	2	3	4	5	6	7	8
11	12	10	9	16	15	13	14	3	4	2	1	8	7	5	6
12	11	9	10	15	16	14	13	4	3	1	2	7	8	6	5
13	14	15	16	10	9	12	11	8	7	5	6	4	3	1	2
14	13	16	15	9	10	11	12	7	8	6	5	3	4	2	1
15	16	14	13	11	12	10	9	6	5	8	7	1	2	3	4
16	15	13	14	12	11	9	10	5	6	7	8	2	1	4	3

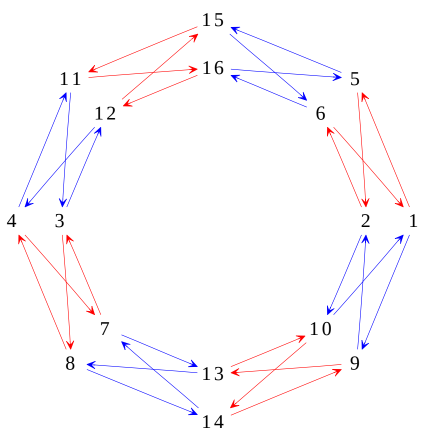

FIG. 2: A Cayley graph of Q_{16}.

representations of dimension 1 or 2 [11, 30, 31]. Table VI shows some of the matrix representations; the notation $e_k = \exp(2\pi i/k)$ for the k-th root of unity is used. Representations of the remaining elements g_k of even index k can be quickly generated from the neighbor g_{k-1} via

$$g_k = g_2 g_{k-1}, \quad (k\,\text{even}, Q_8, Q_{16}, \text{or}\, Q_{32}) \tag{15}$$

as implied by Table V.

Kronecker products involving the 1-dimensional representations do not split (in the spectroscopic sense). They are already block-diagonal upon creation and the CGC can be set to 1:

$$R^{(\alpha)} \otimes R^{(\alpha)} = R^{(1)}, \ (\alpha = 1, 2, 3, 4); \tag{16}$$
$$R^{(2)} \otimes R^{(3)} = R^{(4)}; \quad R^{(2)} \otimes R^{(4)} = R^{(3)}; \tag{17}$$
$$R^{(\alpha)} \otimes R^{(5)} = R^{(5)}, \ (\alpha = 1, 2, 3, 4); \tag{18}$$
$$R^{(2)} \otimes R^{(6)} = R^{(7)}; \quad R^{(2)} \otimes R^{(7)} = R^{(6)}; \tag{19}$$
$$R^{(3)} \otimes R^{(4)} = R^{(2)}; \quad R^{(3)} \otimes R^{(6)} = R^{(7)}; \tag{20}$$
$$R^{(3)} \otimes R^{(7)} = R^{(6)}; \quad R^{(4)} \otimes R^{(6)} = R^{(6)}; \tag{21}$$
$$R^{(4)} \otimes R^{(7)} = R^{(7)}. \tag{22}$$

Kronecker products of 2-dimensional factors split with multiplicities equal to 1:

$$R^{(5)} \otimes R^{(5)} = R^{(1)} \dotplus R^{(2)} \dotplus R^{(3)} \dotplus R^{(4)}; \tag{23}$$
$$R^{(5)} \otimes R^{(6)} = R^{(6)} \dotplus R^{(7)}; \tag{24}$$
$$R^{(5)} \otimes R^{(7)} = R^{(6)} \dotplus R^{(7)}; \tag{25}$$
$$R^{(6)} \otimes R^{(6)} = R^{(1)} \dotplus R^{(4)} \dotplus R^{(5)}; \tag{26}$$
$$R^{(6)} \otimes R^{(7)} = R^{(2)} \dotplus R^{(3)} \dotplus R^{(5)}; \tag{27}$$
$$R^{(7)} \otimes R^{(7)} = R^{(1)} \dotplus R^{(4)} \dotplus R^{(5)}. \tag{28}$$

The 4×4 transformation matrices of (23)–(28) are shown in Tables VII–X. For non-abelian groups, matrix representations may not be symmetric; the freedom of a phase choice remains, but eigenvectors are not necessarily real-valued.

IV. QUATERNION GROUP Q_{32}

The multiplication table of the quaternion group of order $|Q_{32}| = 32$ is Table XI.

The group comprises 11 conjugacy classes:

$$\mathcal{C}_1 = \{g_1\}, \mathcal{C}_2 = \{g_2\}, \mathcal{C}_3 = \{g_3, g_4\}, \mathcal{C}_4 = \{g_5, g_7\}, \tag{29}$$
$$\mathcal{C}_5 = \{g_6, g_8\}, \mathcal{C}_6 = \{g_9, g_{10}, \ldots, g_{16}\}, \tag{30}$$
$$\mathcal{C}_7 = \{g_{17}, g_{18}, \ldots, g_{24}\}, \mathcal{C}_8 = \{g_{25}, g_{29}\}, \tag{31}$$
$$\mathcal{C}_9 = \{g_{26}, g_{30}\}, \mathcal{C}_{10} = \{g_{27}, g_{32}\}, \mathcal{C}_{11} = \{g_{28}, g_{31}\}. \tag{32}$$

The 4 one-dimensional and 7 two-dimensional irreducible representations are partially shown in Table XII. *Partially* means that the representations for the remaining members of the group can be easily calculated with the aid of Table XI. A glance at Figure 3 reveals that following the blue edges and the red edges one can indeed generate all elements starting at g_1; all but the columns labeled 9 and 17 in Table XII are redundant to that task.

TABLE VI: Irreducible representations for a subset of Q_{16} members.

	1	2	3	5	7	9	11	13	15
$R^{(1)}$	1	1	1	1	1	1	1	1	1
$R^{(2)}$	1	1	1	1	1	-1	-1	-1	-1
$R^{(3)}$	1	1	1	-1	-1	1	1	-1	-1
$R^{(4)}$	1	1	1	-1	-1	-1	-1	1	1
$R^{(5)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$R^{(6)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & e_8^3 \\ e_8 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & e_8 \\ e_8^3 & 0 \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$	$\begin{pmatrix} -e_8 & 0 \\ 0 & e_8^3 \end{pmatrix}$
$R^{(7)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -e_8^3 \\ -e_8 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -e_8 \\ -e_8^3 & 0 \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$	$\begin{pmatrix} e_8 & 0 \\ 0 & -e_8^3 \end{pmatrix}$

TABLE VII: CG matrix of $R^{(5)} \otimes R^{(5)}$ of Q_{16} and of Q_{32}.

j	k	$R^{(1)}$	$R^{(2)}$	$R^{(3)}$	$R^{(4)}$
1	1	$1/\sqrt{2}$	$-1/\sqrt{2}$	0	0
1	2	0	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
2	1	0	0	$1/\sqrt{2}$	$1/\sqrt{2}$
2	2	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0

TABLE VIII: CGC of $R^{(5)} \otimes R^{(6)}$ of Q_{16}. The CGC for $R^{(5)} \otimes R^{(7)}$ are the complex-conjugate of these.

		$R^{(6)}$		$R^{(7)}$	
j	k	1	2	1	2
1	1	0	$i/\sqrt{2}$	0	$-i/\sqrt{2}$
1	2	$-i/\sqrt{2}$	0	$i/\sqrt{2}$	0
2	1	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
2	2	$1/\sqrt{2}$	0	$1/\sqrt{2}$	0

TABLE IX: CGC of $R^{(6)} \otimes R^{(6)}$ or $R^{(7)} \otimes R^{(7)}$ of Q_{16}.

		$R^{(1)}$	$R^{(4)}$	$R^{(5)}$	
j	k			1	2
1	1	0	0	$-i\sqrt{2}$	$-1/\sqrt{2}$
1	2	$-1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	1	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	$-i\sqrt{2}$	$1/\sqrt{2}$

TABLE X: CGC of $R^{(6)} \otimes R^{(7)}$ of Q_{16}.

		$R^{(2)}$	$R^{(3)}$	$R^{(5)}$	
j	k			1	2
1	1	0	0	$i\sqrt{2}$	$-1/\sqrt{2}$
1	2	$-1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	1	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	$i\sqrt{2}$	$1/\sqrt{2}$

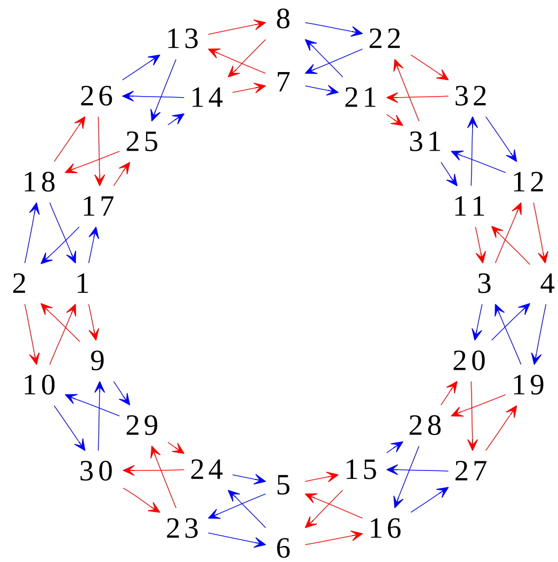

FIG. 3: A Cayley graph of Q_{32}. Blue edges represent multiplication by g_{17} and red edges represent multiplication by g_9.

TABLE XI: Cayley table of Q_{32}, Hall–Senior group 32.51

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23	26	25	28	27	30	29	32	31
3	4	2	1	7	8	6	5	12	11	9	10	16	15	13	14	20	19	17	18	24	23	21	22	27	28	26	25	31	32	30	29
4	3	1	2	8	7	5	6	11	12	10	9	15	16	14	13	19	20	18	17	23	24	22	21	28	27	25	26	32	31	29	30
5	6	7	8	4	3	1	2	15	16	14	13	9	10	11	12	23	24	22	21	17	18	19	20	29	30	31	32	28	27	25	26
6	5	8	7	3	4	2	1	16	15	13	14	10	9	12	11	24	23	21	22	18	17	20	19	30	29	32	31	27	28	26	25
7	8	6	5	1	2	3	4	13	14	15	16	12	11	9	10	21	22	23	24	20	19	17	18	31	32	30	29	25	26	27	28
8	7	5	6	2	1	4	3	14	13	16	15	11	12	10	9	22	21	24	23	19	20	18	17	32	31	29	30	26	25	28	27
9	10	11	12	13	14	15	16	2	1	4	3	6	5	8	7	29	30	31	32	28	27	25	26	24	23	21	22	18	17	20	19
10	9	12	11	14	13	16	15	1	2	3	4	5	6	7	8	30	29	32	31	27	28	26	25	23	24	22	21	17	18	19	20
11	12	10	9	15	16	14	13	3	4	2	1	7	8	6	5	32	31	29	30	26	25	28	27	21	22	23	24	20	19	17	18
12	11	9	10	16	15	13	14	4	3	1	2	8	7	5	6	31	32	30	29	25	26	27	28	22	21	24	23	19	20	18	17
13	14	15	16	12	11	9	10	8	7	5	6	2	1	4	3	25	26	27	28	29	30	31	32	18	17	20	19	22	21	24	23
14	13	16	15	11	12	10	9	7	8	6	5	1	2	3	4	26	25	28	27	30	29	32	31	17	18	19	20	21	22	23	24
15	16	14	13	9	10	11	12	6	5	8	7	3	4	2	1	28	27	25	26	32	31	29	30	20	19	17	18	24	23	21	22
16	15	13	14	10	9	12	11	5	6	7	8	4	3	1	2	27	28	26	25	31	32	30	29	19	20	18	17	23	24	22	21
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
18	17	20	19	22	21	24	23	26	25	28	27	30	29	32	31	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
19	20	18	17	23	24	22	21	28	27	25	26	32	31	29	30	3	4	2	1	7	8	6	5	12	11	9	10	16	15	13	14
20	19	17	18	24	23	21	22	27	28	26	25	31	32	30	29	4	3	1	2	8	7	5	6	11	12	10	9	15	16	14	13
21	22	23	24	20	19	17	18	31	32	30	29	25	26	27	28	8	7	5	6	2	1	4	3	14	13	16	15	11	12	10	9
22	21	24	23	19	20	18	17	32	31	29	30	26	25	28	27	7	8	6	5	1	2	3	4	13	14	15	16	12	11	9	10
23	24	22	21	17	18	19	20	29	30	31	32	28	27	25	26	6	5	8	7	3	4	2	1	16	15	13	14	10	9	12	11
24	23	21	22	18	17	20	19	30	29	32	31	27	28	26	25	5	6	7	8	4	3	1	2	15	16	14	13	9	10	11	12
25	26	27	28	29	30	31	32	18	17	20	19	22	21	24	23	14	13	16	15	11	12	10	9	7	8	6	5	1	2	3	4
26	25	28	27	30	29	32	31	17	18	19	20	21	22	23	24	13	14	15	16	12	11	9	10	8	7	5	6	2	1	4	3
27	28	26	25	31	32	30	29	19	20	18	17	23	24	22	21	15	16	14	13	9	10	11	12	6	5	8	7	3	4	2	1
28	27	25	26	32	31	29	30	20	19	17	18	24	23	21	22	16	15	13	14	10	9	12	11	5	6	7	8	4	3	1	2
29	30	31	32	28	27	25	26	24	23	21	22	18	17	20	19	10	9	12	11	14	13	16	15	1	2	3	4	5	6	7	8
30	29	32	31	27	28	26	25	23	24	22	21	17	18	19	20	9	10	11	12	13	14	15	16	2	1	4	3	6	5	8	7
31	32	30	29	25	26	27	28	22	21	24	23	19	20	18	17	11	12	10	9	15	16	14	13	3	4	2	1	7	8	6	5
32	31	29	30	26	25	28	27	21	22	23	24	20	19	17	18	12	11	9	10	16	15	13	14	4	3	1	2	8	7	5	6

TABLE XII: Irreducible representations for some Q_{32} elements.

	2	3	5	6	9	17	25	26	27	28
$R^{(1)}$	1	1	1	1	1	1	1	1	1	1
$R^{(2)}$	1	1	1	1	1	-1	-1	-1	-1	-1
$R^{(3)}$	1	1	1	1	-1	1	-1	-1	-1	-1
$R^{(4)}$	1	1	1	1	-1	-1	1	1	1	1
$R^{(5)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$R^{(6)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & e_8^3 \\ -e_8 & 0 \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$
$R^{(7)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -e_8^3 \\ e_8 & 0 \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$
$R^{(8)}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & e_{16}^5 \\ e_{16}^3 & 0 \end{pmatrix}$	$\begin{pmatrix} e_{16}^5 & 0 \\ 0 & -e_{16}^3 \end{pmatrix}$	$\begin{pmatrix} -e_{16}^5 & 0 \\ 0 & e_{16}^3 \end{pmatrix}$	$\begin{pmatrix} -e_{16} & 0 \\ 0 & e_{16}^7 \end{pmatrix}$	$\begin{pmatrix} e_{16} & 0 \\ 0 & -e_{16}^7 \end{pmatrix}$
$R^{(9)}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -e_{16}^5 \\ -e_{16}^3 & 0 \end{pmatrix}$	$\begin{pmatrix} -e_{16}^5 & 0 \\ 0 & e_{16}^3 \end{pmatrix}$	$\begin{pmatrix} e_{16}^5 & 0 \\ 0 & -e_{16}^3 \end{pmatrix}$	$\begin{pmatrix} e_{16} & 0 \\ 0 & -e_{16}^7 \end{pmatrix}$	$\begin{pmatrix} -e_{16} & 0 \\ 0 & e_{16}^7 \end{pmatrix}$
$R^{(10)}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -e_{16} \\ -e_{16}^7 & 0 \end{pmatrix}$	$\begin{pmatrix} -e_{16} & 0 \\ 0 & e_{16}^7 \end{pmatrix}$	$\begin{pmatrix} e_{16} & 0 \\ 0 & -e_{16}^7 \end{pmatrix}$	$\begin{pmatrix} -e_{16}^5 & 0 \\ 0 & e_{16}^3 \end{pmatrix}$	$\begin{pmatrix} e_{16}^5 & 0 \\ 0 & -e_{16}^3 \end{pmatrix}$
$R^{(11)}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} -e_8^3 & 0 \\ 0 & e_8 \end{pmatrix}$	$\begin{pmatrix} e_8^3 & 0 \\ 0 & -e_8 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & e_{16} \\ e_{16}^7 & 0 \end{pmatrix}$	$\begin{pmatrix} e_{16} & 0 \\ 0 & -e_{16}^7 \end{pmatrix}$	$\begin{pmatrix} -e_{16} & 0 \\ 0 & e_{16}^7 \end{pmatrix}$	$\begin{pmatrix} e_{16}^5 & 0 \\ 0 & -e_{16}^3 \end{pmatrix}$	$\begin{pmatrix} -e_{16}^5 & 0 \\ 0 & e_{16}^3 \end{pmatrix}$

The non-splitting CG series are:

$$R^{(\alpha)} \otimes R^{(\alpha)} = R^{(1)};\ (\alpha = 2,3,4); \quad (33)$$
$$R^{(2)} \otimes R^{(3)} = R^{(4)}; \quad (34)$$
$$R^{(2)} \otimes R^{(4)} = R^{(3)}; \quad (35)$$
$$R^{(\alpha)} \otimes R^{(5)} = R^{(5)};\ (\alpha = 2,3,4); \quad (36)$$
$$R^{(\alpha)} \otimes R^{(6)} = R^{(7)};\ R^{(\alpha)} \otimes R^{(7)} = R^{(6)};\ (\alpha = 2,3); \quad (37)$$
$$R^{(2)} \otimes R^{(8)} = R^{(9)};\ R^{(2)} \otimes R^{(9)} = R^{(8)}; \quad (38)$$
$$R^{(2)} \otimes R^{(10)} = R^{(11)};\ R^{(2)} \otimes R^{(11)} = R^{(10)}; \quad (39)$$

$$R^{(3)} \otimes R^{(4)} = R^{(2)};\quad R^{(3)} \otimes R^{(8)} = R^{(9)}; \quad (40)$$
$$R^{(3)} \otimes R^{(9)} = R^{(8)};\quad R^{(3)} \otimes R^{(10)} = R^{(11)}; \quad (41)$$
$$R^{(3)} \otimes R^{(11)} = R^{(10)}; \quad (42)$$
$$R^{(4)} \otimes R^{(k)} = R^{(k)};\ (k = 6,7,\ldots,11) \quad (43)$$

The splitting series are:

$$R^{(5)} \otimes R^{(5)} = R^{(1)} \dotplus R^{(2)} \dotplus R^{(3)} \dotplus R^{(4)}; \quad (44)$$
$$R^{(5)} \otimes R^{(6)} = R^{(6)} \dotplus R^{(7)}; \quad (45)$$
$$R^{(5)} \otimes R^{(7)} = R^{(6)} \dotplus R^{(7)}; \quad (46)$$
$$R^{(5)} \otimes R^{(8)} = R^{(10)} \dotplus R^{(11)}; \quad (47)$$
$$R^{(5)} \otimes R^{(9)} = R^{(10)} \dotplus R^{(11)}; \quad (48)$$
$$R^{(5)} \otimes R^{(10)} = R^{(8)} \dotplus R^{(9)}; \quad (49)$$
$$R^{(5)} \otimes R^{(11)} = R^{(8)} \dotplus R^{(9)}; \quad (50)$$

$$R^{(6)} \otimes R^{(6)} = R^{(1)} \dotplus R^{(4)} \dotplus R^{(5)}; \quad (51)$$
$$R^{(6)} \otimes R^{(7)} = R^{(2)} \dotplus R^{(3)} \dotplus R^{(5)}; \quad (52)$$
$$R^{(6)} \otimes R^{(8)} = R^{(8)} \dotplus R^{(11)}; \quad (53)$$
$$R^{(6)} \otimes R^{(9)} = R^{(9)} \dotplus R^{(10)}; \quad (54)$$
$$R^{(6)} \otimes R^{(10)} = R^{(9)} \dotplus R^{(11)}; \quad (55)$$
$$R^{(6)} \otimes R^{(11)} = R^{(8)} \dotplus R^{(10)}; \quad (56)$$

$$R^{(7)} \otimes R^{(7)} = R^{(1)} \dotplus R^{(4)} \dotplus R^{(5)}; \quad (57)$$
$$R^{(7)} \otimes R^{(8)} = R^{(9)} \dotplus R^{(10)}; \quad (58)$$
$$R^{(7)} \otimes R^{(9)} = R^{(8)} \dotplus R^{(11)}; \quad (59)$$
$$R^{(7)} \otimes R^{(10)} = R^{(8)} \dotplus R^{(10)}; \quad (60)$$
$$R^{(7)} \otimes R^{(11)} = R^{(9)} \dotplus R^{(11)}; \quad (61)$$

$$R^{(8)} \otimes R^{(8)} = R^{(1)} \dotplus R^{(4)} \dotplus R^{(6)}; \quad (62)$$
$$R^{(8)} \otimes R^{(9)} = R^{(2)} \dotplus R^{(3)} \dotplus R^{(7)}; \quad (63)$$
$$R^{(8)} \otimes R^{(10)} = R^{(5)} \dotplus R^{(7)}; \quad (64)$$
$$R^{(8)} \otimes R^{(11)} = R^{(5)} \dotplus R^{(6)}; \quad (65)$$

$$R^{(9)} \otimes R^{(9)} = R^{(1)} \dotplus R^{(4)} \dotplus R^{(6)}; \quad (66)$$
$$R^{(9)} \otimes R^{(10)} = R^{(5)} \dotplus R^{(6)}; \quad (67)$$
$$R^{(9)} \otimes R^{(11)} = R^{(5)} \dotplus R^{(7)}; \quad (68)$$

TABLE XIII: CGC of $R^{(5)} \otimes R^{(6)}$ of Q_{32}. The coefficients of $R^{(5)} \otimes R^{(7)}$ are the complex-conjugate of these.

		$R^{(6)}$		$R^{(7)}$	
j	k	1	2	1	2
1	1	0	$i/\sqrt{2}$	0	$-i/\sqrt{2}$
1	2	$i/\sqrt{2}$	0	$-i/\sqrt{2}$	0
2	1	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$
2	2	$-1/\sqrt{2}$	0	$-1/\sqrt{2}$	0

TABLE XIV: CGC of $R^{(5)} \otimes R^{(8)}$ of Q_{32}. The coefficients of $R^{(5)} \otimes R^{(9)}$ are the complex-conjugate of these.

		$R^{(10)}$		$R^{(11)}$	
j	k	1	2	1	2
1	1	$-i/\sqrt{2}$	0	$i/\sqrt{2}$	0
1	2	0	$-i/\sqrt{2}$	0	$i/\sqrt{2}$
2	1	$-1/\sqrt{2}$	0	$-1/\sqrt{2}$	0
2	2	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$

$$R^{(10)} \otimes R^{(10)} = R^{(1)} \dotplus R^{(4)} \dotplus R^{(7)}; \quad (69)$$
$$R^{(10)} \otimes R^{(11)} = R^{(2)} \dotplus R^{(3)} \dotplus R^{(6)}; \quad (70)$$
$$R^{(11)} \otimes R^{(11)} = R^{(1)} \dotplus R^{(4)} \dotplus R^{(7)}. \quad (71)$$

The case (44) is covered by Table VII, the cases (45)–(71) by Tables XIII–XXV.

V. $(C_2 \times D_8) \rtimes C_2$

The factor group of $Q_8 \times Q_8$ with respect to the subgroup containing the unit and `-1` $\times$ `-1` builds another group of order 32, the 49th out of the 51 groups of order 32 in the Small Group Library [11], known as $\Gamma_5 a_1$ [14] and named 32.42 hereafter [15, 16, 32]. It can be assembled by a direct and semi-direct product of the cyclic group C_2 and the dihedral group D_8 [33, 34]. The elements shall be sorted according to the multiplication table XXVI.

If the multiplications with the generators g_3, g_5, g_{11} and g_{17} are coded with blue, red, green and brown edges,

TABLE XV: CGC of $R^{(5)} \otimes R^{(10)}$ of Q_{32}. The coefficients of $R^{(5)} \otimes R^{(11)}$ are the complex-conjugate of these.

		$R^{(8)}$		$R^{(9)}$	
j	k	1	2	1	2
1	1	$i/\sqrt{2}$	0	$-i/\sqrt{2}$	0
1	2	0	$i/\sqrt{2}$	0	$-i/\sqrt{2}$
2	1	$-1/\sqrt{2}$	0	$-1/\sqrt{2}$	0
2	2	0	$1/\sqrt{2}$	0	$1/\sqrt{2}$

TABLE XVI: CG matrix of $R^{(6)} \otimes R^{(6)}$ or $R^{(7)} \otimes R^{(7)}$ of Q_{32}.

		$R^{(1)}$	$R^{(4)}$	$R^{(5)}$	
j	k			1	2
1	1	0	0	$-i/\sqrt{2}$	$-1/\sqrt{2}$
1	2	$1/\sqrt{2}$	$-1/\sqrt{2}$	0	0
2	1	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	$-i/\sqrt{2}$	$1/\sqrt{2}$

TABLE XVII: CGC of $R^{(6)} \otimes R^{(7)}$ of Q_{32}.

		$R^{(2)}$	$R^{(3)}$	$R^{(5)}$	
j	k			1	2
1	1	0	0	$i\sqrt{2}$	$-1/\sqrt{2}$
1	2	$1/\sqrt{2}$	$-1/\sqrt{2}$	0	0
2	1	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	$i\sqrt{2}$	$1/\sqrt{2}$

respectively, the Cayley graph in Fig. 4 results. (These four generators are self-inverse, so the edges are undirected.)

TABLE XVIII: CGC of $R^{(6)} \otimes R^{(8)}$ or $R^{(7)} \otimes R^{(9)}$ of Q_{32}.

		$R^{(8)}$		$R^{(11)}$	
j	k	1	2	1	2
1	1	0	1	0	0
1	2	0	0	-1	0
2	1	0	0	0	1
2	2	-1	0	0	0

TABLE XIX: CGC of $R^{(6)} \otimes R^{(9)}$ or $R^{(7)} \otimes R^{(8)}$ of Q_{32}.

		$R^{(9)}$		$R^{(10)}$	
j	k	1	2	1	2
1	1	0	1	0	0
1	2	0	0	-1	0
2	1	0	0	0	1
2	2	-1	0	0	0

TABLE XX: CGC of $R^{(6)} \otimes R^{(10)}$ or $R^{(7)} \otimes R^{(11)}$ of Q_{32}.

		$R^{(9)}$		$R^{(11)}$	
j	k	1	2	1	2
1	1	0	0	0	1
1	2	-1	0	0	0
2	1	0	1	0	0
2	2	0	0	-1	0

TABLE XXI: CGC of $R^{(6)} \otimes R^{(11)}$ or $R^{(7)} \otimes R^{(10)}$ of Q_{32}.

		$R^{(8)}$		$R^{(10)}$	
j	k	1	2	1	2
1	1	0	0	0	1
1	2	-1	0	0	0
2	1	0	1	0	0
2	2	0	0	-1	0

TABLE XXII: CG matrix of $R^{(8)} \otimes R^{(8)}$ or $R^{(9)} \otimes R^{(9)}$ of Q_{32}. Replacing $R^{(6)}$ by $R^{(7)}$ provides the table for $R^{(10)} \otimes R^{(10)}$ and $R^{(11)} \otimes R^{(10)}$.

		$R^{(1)}$	$R^{(4)}$	$R^{(6)}$	
j	k			1	2
1	1	0	0	0	1
1	2	$-1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	1	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	1	0

TABLE XXIII: CG matrix of $R^{(8)} \otimes R^{(9)}$ of Q_{32}. Replacing $R^{(7)}$ by $R^{(6)}$ provides the table for $R^{(10)} \otimes R^{(11)}$.

		$R^{(2)}$	$R^{(3)}$	$R^{(7)}$	
j	k			1	2
1	1	0	0	0	1
1	2	$-1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	1	$1/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	1	0

TABLE XXIV: CGC of $R^{(8)} \otimes R^{(10)}$ or $R^{(9)} \otimes R^{(11)}$ of Q_{32}.

		$R^{(5)}$		$R^{(7)}$	
j	k	1	2	1	2
1	1	0	0	1	0
1	2	$i/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	1	$-i/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	0	1

TABLE XXV: CGC of $R^{(8)} \otimes R^{(11)}$ or $R^{(9)} \otimes R^{(10)}$ of Q_{32}.

		$R^{(5)}$		$R^{(6)}$	
j	k	1	2	1	2
1	1	0	0	1	0
1	2	$-i/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	1	$i/\sqrt{2}$	$1/\sqrt{2}$	0	0
2	2	0	0	0	1

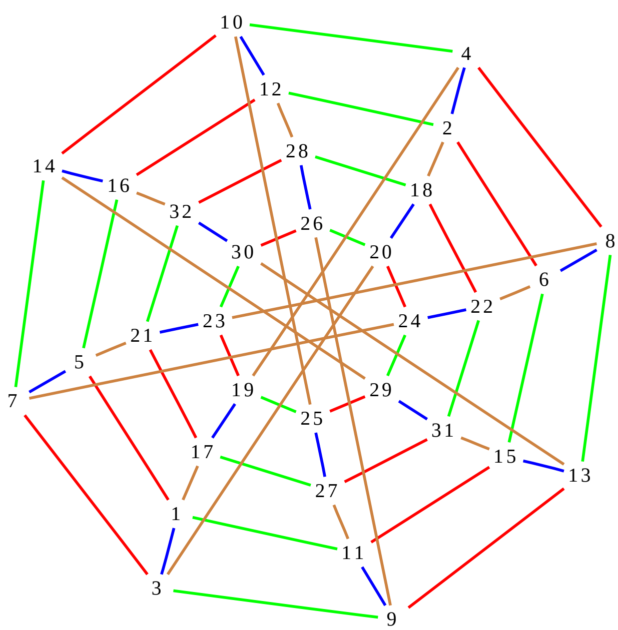

FIG. 4: A Cayley graph of 32.42.

TABLE XXVI: Cayley table of 32.42

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23	26	25	28	27	30	29	32	31
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14	20	19	18	17	24	23	22	21	28	27	26	25	32	31	30	29
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13	19	20	17	18	23	24	21	22	27	28	25	26	31	32	29	30
5	6	7	8	1	2	3	4	14	13	16	15	10	9	12	11	21	22	23	24	17	18	19	20	30	29	32	31	26	25	28	27
6	5	8	7	2	1	4	3	13	14	15	16	9	10	11	12	22	21	24	23	18	17	20	19	29	30	31	32	25	26	27	28
7	8	5	6	3	4	1	2	16	15	14	13	12	11	10	9	24	23	22	21	20	19	18	17	31	32	29	30	27	28	25	26
8	7	6	5	4	3	2	1	15	16	13	14	11	12	9	10	23	24	21	22	19	20	17	18	32	31	30	29	28	27	26	25
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8	26	25	28	27	30	29	32	31	18	17	20	19	22	21	24	23
10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7	25	26	27	28	29	30	31	32	17	18	19	20	21	22	23	24
11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6	27	28	25	26	31	32	29	30	19	20	17	18	23	24	21	22
12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5	28	27	26	25	32	31	30	29	20	19	18	17	24	23	22	21
13	14	15	16	9	10	11	12	6	5	8	7	2	1	4	3	30	29	32	31	26	25	28	27	21	22	23	24	17	18	19	20
14	13	16	15	10	9	12	11	5	6	7	8	1	2	3	4	29	30	31	32	25	26	27	28	22	21	24	23	18	17	20	19
15	16	13	14	11	12	9	10	8	7	6	5	4	3	2	1	31	32	29	30	27	28	25	26	24	23	22	21	20	19	18	17
16	15	14	13	12	11	10	9	7	8	5	6	3	4	1	2	32	31	30	29	28	27	26	25	23	24	21	22	19	20	17	18
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
18	17	20	19	22	21	24	23	26	25	28	27	30	29	32	31	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
19	20	17	18	23	24	21	22	27	28	25	26	31	32	29	30	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
20	19	18	17	24	23	22	21	28	27	26	25	32	31	30	29	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
21	22	23	24	17	18	19	20	30	29	32	31	26	25	28	27	5	6	7	8	1	2	3	4	14	13	16	15	10	9	12	11
22	21	24	23	18	17	20	19	29	30	31	32	25	26	27	28	6	5	8	7	2	1	4	3	13	14	15	16	9	10	11	12
23	24	21	22	19	20	17	18	32	31	30	29	28	27	26	25	8	7	6	5	4	3	2	1	15	16	13	14	11	12	9	10
24	23	22	21	20	19	18	17	31	32	29	30	27	28	25	26	7	8	5	6	3	4	1	2	16	15	14	13	12	11	10	9
25	26	27	28	29	30	31	32	17	18	19	20	21	22	23	24	10	9	12	11	14	13	16	15	2	1	4	3	6	5	8	7
26	25	28	27	30	29	32	31	18	17	20	19	22	21	24	23	9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
27	28	25	26	31	32	29	30	19	20	17	18	23	24	21	22	11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
28	27	26	25	32	31	30	29	20	19	18	17	24	23	22	21	12	11	10	9	16	15	14	13	4	3	2	1	8	7	6	5
29	30	31	32	25	26	27	28	22	21	24	23	18	17	20	19	14	13	16	15	10	9	12	11	5	6	7	8	1	2	3	4
30	29	32	31	26	25	28	27	21	22	23	24	17	18	19	20	13	14	15	16	9	10	11	12	6	5	8	7	2	1	4	3
31	32	29	30	27	28	25	26	24	23	22	21	20	19	18	17	15	16	13	14	11	12	9	10	8	7	6	5	4	3	2	1
32	31	30	29	28	27	26	25	23	24	21	22	19	20	17	18	16	15	14	13	12	11	10	9	7	8	5	6	3	4	1	2

TABLE XXVII: Characters of the 32.42 classes (73).

$\backslash\mathcal{C}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$R^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$R^{(2)}$	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
$R^{(3)}$	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
$R^{(4)}$	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
$R^{(5)}$	1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1
$R^{(6)}$	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
$R^{(7)}$	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
$R^{(8)}$	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
$R^{(9)}$	1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
$R^{(10)}$	1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1
$R^{(11)}$	1	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
$R^{(12)}$	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
$R^{(13)}$	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
$R^{(14)}$	1	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
$R^{(15)}$	1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1	1	-1
$R^{(16)}$	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1	1	-1	-1	1
$R^{(17)}$	4	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

With this enumeration of the elements, the 17 conjugacy classes are

$$\mathcal{C}_1 = \{g_1\}, \mathcal{C}_2 = \{g_2\}, \tag{72}$$

$$\mathcal{C}_j = \{g_{2j-3}, g_{2j-2}\}, (j = 3, 4, \ldots, 17). \tag{73}$$

The 16 one-dimensional representations are part of Table XXVII. The 4-dimensional representation is the unit and negative unit matrix for g_1 and g_2. A possible set of the other elements is written with the aid of 2×2 submatrices

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \tag{74}$$

$$\tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{75}$$

as

$$g_3 = \begin{pmatrix} \tau_0 & 0 \\ 0 & -\tau_0 \end{pmatrix}; g_5 = \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix} \tag{76}$$

$$g_7 = \begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix}; g_9 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}; \tag{77}$$

$$g_{11} = \begin{pmatrix} \tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix}; g_{13} = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}; \tag{78}$$

$$g_{15} = \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix}; g_{17} = \begin{pmatrix} 0 & \tau_3 \\ \tau_3 & 0 \end{pmatrix}; \tag{79}$$

$$g_{19} = \begin{pmatrix} 0 & -\tau_3 \\ \tau_3 & 0 \end{pmatrix}; g_{21} = \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{pmatrix}; \tag{80}$$

$$g_{23} = \begin{pmatrix} 0 & -\tau_0 \\ \tau_0 & 0 \end{pmatrix}; g_{25} = \begin{pmatrix} 0 & -\tau_2 \\ -\tau_2 & 0 \end{pmatrix}; \tag{81}$$

$$g_{27} = \begin{pmatrix} 0 & \tau_2 \\ -\tau_2 & 0 \end{pmatrix}; g_{29} = \begin{pmatrix} 0 & -\tau_1 \\ -\tau_1 & 0 \end{pmatrix}; \tag{82}$$

$$g_{31} = \begin{pmatrix} 0 & \tau_1 \\ -\tau_1 & 0 \end{pmatrix}. \tag{83}$$

All remaining representations are available by multiplication with g_2 as indicated by Table XXVI and Eq. (15).

The Kronecker products of pairs of one-dimensional representations are not discussed in detail, because this reduces to multiplying two rows containing a string of ± 1 in the character table XXVII and finding the row that matches that binary pattern. The cases

$$R^{(\alpha)} \otimes R^{(17)} = R^{(17)}, (\alpha = 1, \ldots, 16) \tag{84}$$

do not split either. The remaining case is

$$R^{(17)} \otimes R^{(17)} = R^{(1)} \dotplus R^{(2)} \dotplus R^{(3)} \dotplus \cdots \dotplus R^{(16)}. \tag{85}$$

Its associated CG matrix defines Table XXVIII.

VI. SUMMARY

The Clebsch–Gordan coefficients of four finite non-abelian groups with irreducible representations of dimension up to $n_\alpha = 4$ have been computed by direct diagonalization of matrices of dimension n_α^4.

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TABLE XXVIII: CGC of $R^{(17)} \otimes R^{(17)}$ of 32.42.

j	k	$R^{(1)}$	$R^{(2)}$	$R^{(3)}$	$R^{(4)}$	$R^{(5)}$	$R^{(6)}$	$R^{(7)}$	$R^{(8)}$	$R^{(9)}$	$R^{(10)}$	$R^{(11)}$	$R^{(12)}$	$R^{(13)}$	$R^{(14)}$	$R^{(15)}$	$R^{(16)}$
1	1	1/2	−1/2	−1/2	1/2	0	0	0	0	0	0	0	0	0	0	0	0
1	2	0	0	0	0	−1/2	1/2	1/2	−1/2	0	0	0	0	0	0	0	0
1	3	0	0	0	0	0	0	0	0	1/2	−1/2	−1/2	1/2	0	0	0	0
1	4	0	0	0	0	0	0	0	0	0	0	0	0	−1/2	1/2	1/2	−1/2
2	1	0	0	0	0	−1/2	1/2	−1/2	1/2	0	0	0	0	0	0	0	0
2	2	1/2	−1/2	1/2	−1/2	0	0	0	0	0	0	0	0	0	0	0	0
2	3	0	0	0	0	0	0	0	0	0	0	0	0	−1/2	1/2	−1/2	1/2
2	4	0	0	0	0	0	0	0	0	1/2	−1/2	1/2	−1/2	0	0	0	0
3	1	0	0	0	0	0	0	0	0	1/2	1/2	−1/2	−1/2	0	0	0	0
3	2	0	0	0	0	0	0	0	0	0	0	0	0	1/2	1/2	−1/2	−1/2
3	3	1/2	1/2	−1/2	−1/2	0	0	0	0	0	0	0	0	0	0	0	0
3	4	0	0	0	0	1/2	1/2	−1/2	−1/2	0	0	0	0	0	0	0	0
4	1	0	0	0	0	0	0	0	0	0	0	0	0	1/2	1/2	1/2	1/2
4	2	0	0	0	0	0	0	0	0	1/2	1/2	1/2	1/2	0	0	0	0
4	3	0	0	0	0	1/2	1/2	1/2	1/2	0	0	0	0	0	0	0	0
4	4	1/2	1/2	1/2	1/2	0	0	0	0	0	0	0	0	0	0	0	0

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